MATH 1650 LOGARITHMIC FUNCTIONS WORKSHEET

Exponential functions $f(x) = b^x$ are are one-to-one which means they are invertible. In this section, we explore their inverses, the *logarithmic functions* which are called 'logs' for short. In other words...

$b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.

EXAMPLE: Find the exact value of the logarithms below:

- $\log_3(81)$ is the exponent you put on 3 to get 81 . $3^? = 81 = 3^4$.

  Hence, $\log_3(81) = 4$.

- $\log_2\left(\frac{1}{8}\right)$ is the exponent you put on 2 to get $\frac{1}{8}$. $2^? = \frac{1}{8} = 2^{-3}$.

  Hence, $\log_2\left(\frac{1}{8}\right) = -3$.

- $\log_{25}(5)$ is the exponent you put on 25 to get 5 . $25^? = 5 = \sqrt{25} = (25)^{\frac{1}{2}}$.

  Hence, $\log_{25}(5) = \frac{1}{2}$.

- **NOTE:** '$\log_{10}(x)$' is usually written as '$\log(x)$.'

  $\log(1000)$ is the exponent you put on 10 to get 1000 . $10^? = 1000 = 10^3$.

  Hence, $\log(1000) = 3$.

- **NOTE:** '$\log_e(x)$' is usually written as '$\ln(x)$.'

  $\ln(\sqrt{e^3})$ is the exponent you put on 3 to get $\sqrt{e^3}$. $e^? = \sqrt{e^3} = e^{\frac{3}{2}}$

  Hence, $\ln(\sqrt{e^3}) = \frac{3}{2}$.

- $\log_2(2^{1.3})$ is the exponent you put on 2 to get $2^{1.3}$. $2^? = 2^{1.3}$.

  Hence, $\log_2(2^{1.3}) = 1.3$.

  NOTE: In general, $\log_b(b^x) = x$ for all real numbers. Do you see why?

- $\log_3(5)$ is the exponent you put on 3 to get 5 . $3^? = 5$.

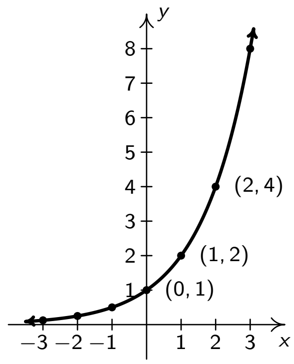
  Hence, $3^{\log_3(5)} = 5$.

  NOTE: In general, $b^{\log_b(x)} = x$ for all $x > 0$. Do you see why?

EXAMPLE:

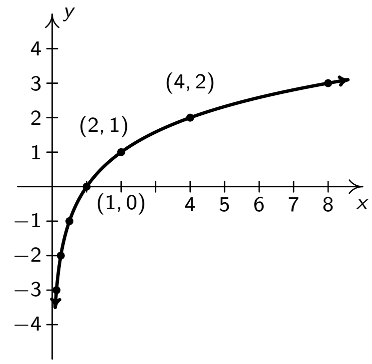
- Graph $y = \log_2(x)$ by starting with $y = 2^x$.

  Label three points on each graph and the asymptotes.



Horizontal Asymptote $y = 0$

$y = 2^x$

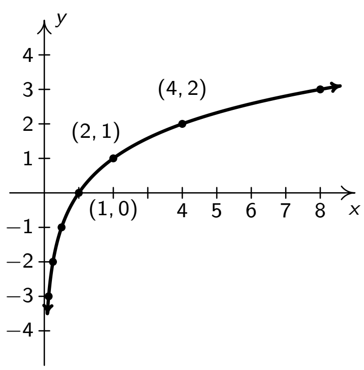


Vertical Asymptote $x = 0$

$y = \log_2(x)$

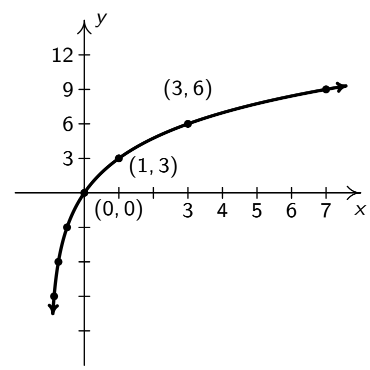
- Graph $f(x) = 3\log_2(x+1)$ by starting with $y = \log_2(x)$ and using transformations.

  Label three points on each graph and the asymptotes.



Vertical Asymptote $x = 0$

$y = \log_2(x)$



Vertical Asymptote $x = -1$

$f(x) = 3\log_2(x+1)$

EXAMPLE:

- $\log_2(-1)$ is asking $2^? = -1$.

  Since $2^x > 0$ for all real numbers x, there is no real number solution to $2^? = -1$.

- $\log_2(0)$ is asking $2^? = 0$.

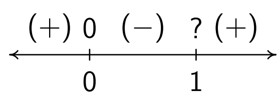
  Since $2^x > 0$ for all real numbers x, there is no real number solution to $2^? = 0$.

EXAMPLE:

- To find the domain of $f(x) = 2\log(3-x) - 1$, we set $3 - x > 0$ to obtain $x < 3$, or $(-\infty, 3)$.

- To find the domain of $g(x) = \ln\left(\dfrac{x}{x-1}\right)$, we solve $\dfrac{x}{x-1} > 0$ using a Sign Diagram.

  If we define $r(x) = \dfrac{x}{x-1}$, we find r is undefined at $x = 1$ and $r(x) = 0$ when $x = 0$. We get:

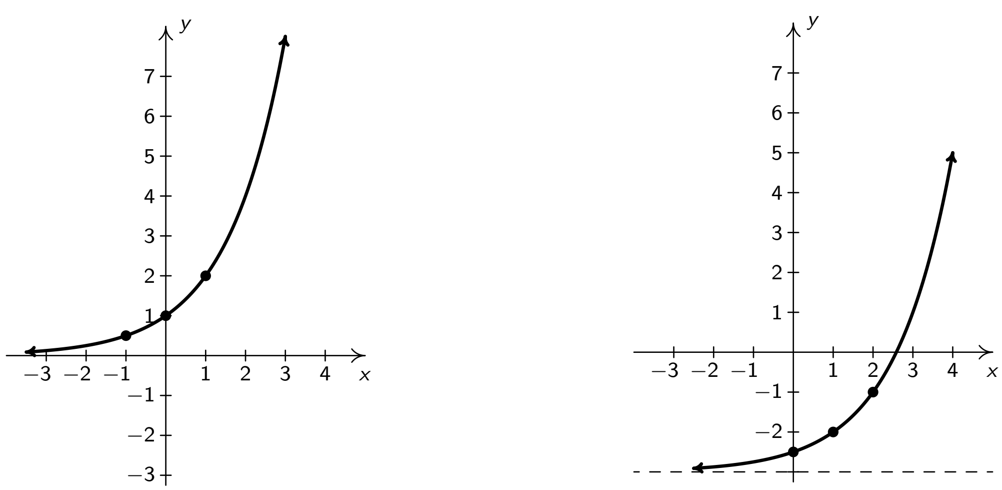


  We find $\dfrac{x}{x-1} > 0$ on $(-\infty, 0) \cup (1, \infty)$ which is the domain of g.

EXAMPLE:

- We graph $f(x) = 2^{x-1} - 3$ by starting with $y = 2^x$ and using transformations:

  Namely, we first move to the right 1 unit and then down three units:

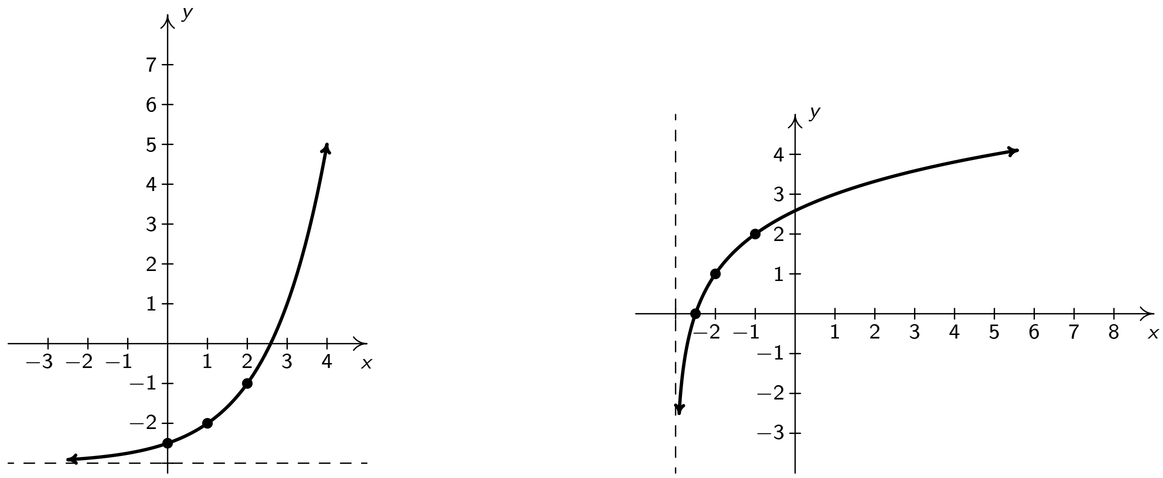


$y = 2^x$ $\longrightarrow$ $y = f(x) = 2^{x-1} - 3$

  From the graph of f, we get the domain is $(-\infty, \infty)$ and the range is $(-3, \infty)$.

- f is invertible since the graph of f passes the Horizontal Line Test.

  To graph f^{-1}, we interchange the x and y coordinates on the points of the graph of f.



$y = f(x) = 2^{x-1} - 3$ $\longrightarrow$ $y = f^{-1}(x) = \log_2(x+3) + 1$

  The domain of f^{-1} is $(-3, \infty)$ (which matches the range of f).

  The range of f^{-1} is $(-\infty, \infty)$ (which matches the domain of f.)

- Thinking of f as a process, the formula $f(x) = 2^{x-1} - 3$ takes an input x and applies the steps:

  1. subtract 1.
  2. put the result of the first step as the exponent on 2.
  3. subtract 3 from the result of the second step.

  Clearly, to undo subtracting 1, we will add 1, and similarly we undo subtracting 3 by adding 3. How do we undo the second step? The answer is we use the logarithm.

  By definition, $\log_2(x)$ undoes exponentiation by 2. Hence, f^{-1} should:

  1. add 3.
  2. take the logarithm base 2 of the result of the first step.
  3. add 1 to the result of the second step.

  In symbols, $f^{-1}(x) = \log_2(x+3) + 1$.

- When simplifying $(f^{-1} \circ f)(x)$ we assume x can be any real number.

  When simplifying $(f \circ f^{-1})(x)$, we restrict our attention to $x > -3$. (Do you see why?)

$$\begin{aligned}
\left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\
&= f^{-1}\left(2^{x-1} - 3\right) \\
&= \log_2\left(\left[2^{x-1} - 3\right] + 3\right) + 1 \\
&= \log_2\left(2^{x-1}\right) + 1 \\
&= (x-1) + 1 \\
&= x \checkmark
\end{aligned}
\qquad
\begin{aligned}
\left(f \circ f^{-1}\right)(x) &= f\left(f^{-1}(x)\right) \\
&= f\left(\log_2(x+3) + 1\right) \\
&= 2^{(\log_2(x+3)+1)-1} - 3 \\
&= 2^{\log_2(x+3)} - 3 \\
&= (x+3) - 3 \\
&= x \checkmark
\end{aligned}$$

- Viewing $2^{x-1} - 3 = 4$ as $f(x) = 4$, we apply f^{-1} to 'undo' f to get $f^{-1}(f(x)) = f^{-1}(4)$ or $x = f^{-1}(4)$. Since $f^{-1}(x) = \log_2(x+3) + 1$, we get $x = f^{-1}(4) = \log_2(7) + 1$.